Homework 3, due February 5, 1999

Problem 1. Ashcroft-Mermin 1.3

Assume a collision at $\vec{r}_1$ and the next collision at $\vec{r}_2$. The change in energy is

$\Delta\epsilon = \frac{1}{2}m(\vec{v}_1 - \frac{e\vec{E}}{m}t)^2 - \frac{1}{2}m(\vec{v}_2)^2$

where $\vec{v}_i$ is the velocity just after collision i. Thermal averages give us:

$\epsilon(T(\vec{r}_1)) =< \frac{1}{2}m(\vec{v}_1)^2 >$

$\epsilon(T(\vec{r}_2)) =< \frac{1}{2}m(\vec{v}_2)^2 >$

$< \vec{v}_i >= 0$

and hence

$< \Delta\epsilon >=< \epsilon(T(\vec{r}_1)) - \epsilon(T(\vec{r}_2)) > + \frac{e^2E^2}{2m} < t^2 >$

The last term is simple Joule heating. There is still an average in the first part, since we have to find the difference in position. The first term gives a contribution

$< \Delta\epsilon >= \frac{d\epsilon}{dT}(T_{av}) < (T(\vec{r}_1) - T(\vec{r}_2)) >$

where we assume that the temperature gradients are small and can take the derivative at the average temperature of the sample.

We have

$\vec{r}_2 - \vec{r}_1 = \vec{v}_1 t - \frac{e\vec{E}}{2m}t^2$

and hence

$< \vec{r}_2 - \vec{r}_1 >= -\frac{e\vec{E}}{2m}2\tau^2$

This gives

$< (T(\vec{r}_1) - T(\vec{r}_2)) >= \vec{\nabla}T \cdot < \vec{r}_1 - \vec{r}_2 >= \vec{\nabla}T \cdot \vec{E}\frac{e}{m}\tau^2$

and the average energy loss per colision is

$\Delta\epsilon = \frac{d\epsilon}{dT}(T_{av})\vec{\nabla}T \cdot \vec{E}\frac{e}{m}\tau^2$

The power loss (or power generated in heat) is

$P = n\frac{\Delta\epsilon}{\tau} = \frac{d\epsilon}{dT}(T_{av})\vec{\nabla}T \cdot \vec{E}\frac{ne\tau}{m} = \frac{d\epsilon}{dT}(T_{av})\vec{\nabla}T \cdot \vec{E}\frac{\sigma_0}{e}$

or

$P = \frac{d\epsilon}{dT}(T_{av})\vec{\nabla}T \cdot \frac{\vec{j}}{e}$

Problem 1. Ashcroft-Mermin 1.4

(a) From $\frac{d\vec{p}}{dt} = -\frac{1}{\tau}\vec{p} + \vec{f}$ assuming that we only have one Fourier component we get

$-i\omega\vec{p}(t) = -\frac{1}{\tau}\vec{p}(t) - e\vec{E}(t) - \frac{e}{mc}\vec{p}(t) \times \vec{H}$

Writing $\vec{p}(t) = (p_x, p_y, p_z)e^{-i\omega t}$ and similarly $\vec{E}(t) = (E_x, E_y, E_z)e^{-i\omega t}$ we get

$(\frac{1}{\tau} - i\omega)p_x = -eE_x - \omega_C p_y$

$(\frac{1}{\tau} - i\omega)p_y = -eE_y + \omega_C p_x$

$(\frac{1}{\tau} - i\omega)p_z = 0$

The last equation gives $p_z = 0$ and hence $j_z = 0$

The first two equations give

$$\begin{pmatrix} \frac{1}{\tau} - i\omega & \omega_C \\ -\omega_C & \frac{1}{\tau} - i\omega \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = -e \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

with the inverse equation

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = -e \frac{1}{(\frac{1}{\tau} - i\omega)^2 + \omega_C^2} \begin{pmatrix} \frac{1}{\tau} - i\omega & -\omega_C \\ \omega_C & \frac{1}{\tau} - i\omega \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

Using $E_y = \pm i E_x$ this gives

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = -eE_x \frac{1}{(\frac{1}{\tau} - i\omega)^2 + \omega_C^2} \begin{pmatrix} \frac{1}{\tau} - i\omega & -\omega_C \\ \omega_C & \frac{1}{\tau} - i\omega \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

or

$$p_x = -eE_x \frac{1}{(\frac{1}{\tau} - i\omega)^2 + \omega_C^2} (\frac{1}{\tau} - i\omega \mp i\omega_C)$$

$$p_y = -eE_x \frac{1}{(\frac{1}{\tau} - i\omega)^2 + \omega_C^2} (-\omega_C \mp i(\frac{1}{\tau} - i\omega)) = \pm i p_x$$

which shows $j_y = \pm i j_x$ and also

$$j_x = -\frac{ne}{m} p_x = \frac{ne^2}{m} E_x \frac{1}{(\frac{1}{\tau} - i\omega)^2 + \omega_C^2} (\frac{1}{\tau} - i\omega \mp i\omega_C)$$

This is equal to

$$j_x = \frac{ne^2}{m} E_x \frac{1}{\frac{1}{\tau} - i\omega \pm i\omega_C} = \frac{ne^2\tau}{m} E_x \frac{1}{1 - i(\omega \mp \omega_C)\tau}$$

(b) With $\Delta\vec{E} = -\frac{\omega^2}{c^2}\epsilon(\omega)\vec{E}$ and $\vec{E} = E_x(1, \pm i, 0)e^{i(kz - \omega t)}$ we have a solution as long as

$$k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)$$

Also formula 1.32 tells us

$$\Delta\vec{E} = -\frac{i\omega}{c}(\frac{4\pi}{c}\vec{j} - \frac{i\omega}{c}\vec{E})$$

or

$$\frac{\omega^2}{c^2}\epsilon(\omega) = \frac{i\omega}{c}(\frac{4\pi}{c}\sigma_0 \frac{1}{1 - i(\omega \mp \omega_C)\tau} - \frac{i\omega}{c})$$

which gives

$$\epsilon(\omega) = 1 + \frac{4i\pi}{\omega}\sigma_0 \frac{1}{1 - i(\omega \mp \omega_C)\tau} = 1 - \frac{4\pi}{\omega\tau}\sigma_0 \frac{1}{\frac{i}{\tau} + (\omega \mp \omega_C)} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega}{\frac{i}{\tau} + (\omega \mp \omega_C)}$$

This is like the formula we had before, but with a correction term $\frac{\omega}{\frac{i}{\tau} + (\omega \mp \omega_C)}$

(c) Express the frequency in units of ω_C, using $x = \frac{\omega}{\omega_C}$. Then we have (with the plus sign for the field, hence the minus sign in the equation!)

$$\epsilon(x) = 1 - \frac{1}{x^2} \frac{\omega_p^2}{\omega_C^2} \frac{x}{\frac{i}{\omega_C\tau} + (x - 1)}$$

For a typical value take $\omega_C\tau = 100$ and $\frac{\omega_p^2}{\omega_C^2} = 10^6$, which yields

$\epsilon(x) = 1 - \frac{1}{x^2}\frac{x10^{12}}{0.01i+(x-1)}$

with real and imaginary parts:

$\Re\epsilon(x) = 1 - \frac{1}{x^2}\frac{x10^{12}(x-1)}{10^{-4}+(x-1)^2}$

$\Im\epsilon(x) = \frac{1}{x^2}\frac{x10^{10}}{10^{-4}+(x-1)^2}$

There are two regimes:

(A) $|x-1| >> \frac{1}{\omega_c\tau}$ for which we have

$\Re\epsilon(x) \approx 1 - \frac{1}{x^2}\frac{x10^{12}}{(x-1)}$

$\Im\epsilon(x) \approx \frac{1}{x^2}\frac{x10^{10}}{(x-1)^2}$

For $x >> 1$ this gives $\Re\epsilon(x) \approx 1 - \frac{10^{12}}{x^2}$ $\Im\epsilon(x) \approx \frac{10^{10}}{x^3}$ with the normal plasma oscillation at $x = 10^6$.

For $0 < x << 1$ we have $\Re\epsilon(x) \approx \frac{10^{12}}{x}$ $\Im\epsilon(x) \approx \frac{10^{10}}{x}$ and unlike before this now diverges to plus infinity.

(B) $|x-1| << \frac{1}{\omega_c\tau}$ for which we have

$\Re\epsilon(x) = 1 - 10^{16}(x-1)$

$\Im\epsilon(x) = 10^{14}$

which is a standard Lorentzian form with a resonance at x=1. In this case the electric field is in resonance with the magnetic cyclotron frequency and strong absorption is observed.

Finally, there are always solutions for $k^2c^2 = \Re\epsilon\omega^2$ both for $\omega < \omega_c$ and $\omega > \omega_p$. The second case is the easiest. For $\omega > \omega_p$ we have $\epsilon \approx 1 - \frac{\omega_p^2}{\omega^2}$ and hence need to solve $k^2c^2 = \omega^2 - \omega_p^2$ which always has a solution. These is the case discussed in the book, the electric field is not absorbed and the magnetic field does not influence this regime. The first case is new, and here the effect of the magnetic field is essential. We have $\Re\epsilon \approx 100\Im\epsilon$ and absorption is small. From $\Re\epsilon \approx \frac{\omega_p^2}{\omega\omega_C}$ we need to solve $k^2c^2 = \omega\frac{\omega_p^2}{\omega_C}$ which again always has a solution. These are the helicon waves.

Problem 1. Ashcroft-Mermin 1.5

(a) For $z > 0$ we have $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow iqE_x = KE_z$ or $iqA = KB$

Similarly for $z < 0$ $iqC = -K'D$

For $z > 0$ we have $\Delta\vec{E} = (-q^2 + K^2)\vec{E} = -\frac{\omega^2}{c^2}\epsilon\vec{E} \Rightarrow K^2 - q^2 = -\frac{\omega^2}{c^2}\epsilon$

Similarly for $z < 0$ $K'^2 - q^2 = -\frac{\omega^2}{c^2}$

Finally the continuity gives for E_x $A = C$ and for E_z $\epsilon B = D$

The first two equation give $\frac{K}{K'} = \frac{iqA}{B}\frac{-D}{iqC} = -\frac{AD}{BC} = -\epsilon$ where we used the continuity equations in the last step.

Subtracting the third and fourth equation yields $K^2 - K'^2 = \frac{\omega^2}{c^2}(1-\epsilon)$ and using the relation between K and K' we get $K^2(1-\frac{1}{\epsilon^2}) = \frac{\omega^2}{c^2}(1-\epsilon)$

This gives $K^2 = -\left(\frac{\omega\epsilon}{c}\right)^2 \frac{1}{\epsilon+1}$ and $K'^2 = -\left(\frac{\omega}{c}\right)^2 \frac{1}{\epsilon+1}$

In order for a solution to exist we need $\epsilon < -1$ and hence we need to be below the bulk plasmon frequency. This gives

$K = -\frac{\omega\epsilon}{c}\sqrt{\frac{-1}{\epsilon+1}}$ and $K' = \frac{\omega}{c}\sqrt{\frac{-1}{\epsilon+1}}$

since we need $K > 0$ and $K' > 0$

Finally, $q^2 = K^2 + \frac{\omega^2}{c^2}\epsilon = \frac{\omega^2}{c^2}\frac{\epsilon}{1+\epsilon}$ which is positive indeed.

(b) If $\omega\tau >> 1$ we have $\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$ and hence $q^2c^2 = \omega^2 \frac{1-\frac{\omega_p^2}{\omega^2}}{2-\frac{\omega_p^2}{\omega^2}}$

or

$q^2c^2 = \omega^2 \frac{\omega^2-\omega_p^2}{2\omega^2-\omega_p^2}$

which is easy to plot (assymptote at $2\omega^2 = \omega_p^2$).

(c) If $qc >> \omega$ the second factor needs to be large and we need $\omega^2 = \frac{1}{2}\omega_p^2 - \delta$ for a solution, with δ small. This gives $q^2c^2 = \frac{1}{8\delta}\omega_p^4$ or $\delta = \frac{1}{8q^2c^2}\omega_p^4$

also

$K' = \frac{\omega}{c}\sqrt{\frac{-\omega^2}{2\omega^2-\omega_p^2}} = \frac{\omega_p^2}{c2\sqrt{2\delta}}$ becomes very large, and also K becomes very large. Hence the solution is localized at the surface.

Using the solution for δ we get $K' = q$ and hence $D = -iC$. The wave is circularly polarized in vacuum and has elliptic polarization in the metal.