PHYSICS 880.06 (Fall 2005) Proble Set 1 Solution

(1.1) A&M Problem 1.4

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= -e\left(\mathbf{E} + \frac{\mathbf{p}}{mc} \times \mathbf{H}\right) - \frac{\mathbf{p}}{\tau}, \\
\mathbf{H} &= H_z \hat{\mathbf{z}}, \\
\mathbf{E}(t) &= \mathrm{Re}(\mathbf{E}(\omega) e^{-i\omega t}).
\end{aligned}
$$

(a) Seek steady-state solutoin of this form

$$
\begin{aligned}
\mathbf{p}(t) &= \mathrm{Re}(\mathbf{p}(\omega) e^{-i\omega t}), \\
-i\omega \mathbf{p}(\omega) &= -e\left(\mathbf{E}(\omega) + \frac{\mathbf{p}(\omega)}{mc} \times \mathbf{H}\right) - \frac{\mathbf{p}(\omega)}{\tau}.
\end{aligned}
$$

$$
\begin{aligned}
\left(-i\omega + \frac{1}{\tau}\right) p_x(\omega) &= -e\left(E_x(\omega) + \frac{1}{mc} p_y(\omega) H_z\right), \\
\left(-i\omega + \frac{1}{\tau}\right) p_y(\omega) &= -e\left(E_y(\omega) - \frac{1}{mc} p_x(\omega) H_z\right), \\
\left(-i\omega + \frac{1}{\tau}\right) p_z(\omega) &= -e E_z(\omega).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(\omega) &= E_x(\omega)\hat{\mathbf{x}} + E_y(\omega)\hat{\mathbf{y}}, \\
E_y &= \pm i E_x, \\
E_z &= 0.
\end{aligned}
$$

The solution is

$$
\begin{aligned}
p_x &= \frac{-e\tau}{1 - i(\omega \mp \omega_c)\tau} E_x, \\
p_y &= \pm i p_x, \\
p_z &= 0,
\end{aligned}
$$

where

$$
\omega_c = \frac{eH_z}{mc}.
$$

The current density is

$$
\begin{aligned}
\mathbf{j} &= -ne\frac{\mathbf{p}}{m}, \\
j_x &= \frac{\sigma_0}{1 - i(\omega \mp \omega_c)\tau} E_x, \\
j_y &= \pm i j_x, \\
j_z &= 0,
\end{aligned}
$$

where

$$
\sigma_0 = \frac{ne^2\tau}{m}.
$$

(b) From Maxwell equations,

$$-\nabla^2 \mathbf{E} = \frac{\omega^2}{c^2}\left(1+\frac{4\pi i\sigma}{\omega}\right)\mathbf{E},$$
$$\sigma_{xx}=\sigma_{yy} = \frac{\sigma_0}{1-i(\omega\mp\omega_c)\tau}.$$

Look for a solution of this form $E_x(k,t)=E_0e^{-i(kz-\omega t)}$. Plugging in,

$$k^2c^2 = \omega^2\left(1-\frac{\omega_p^2}{\omega}\frac{1}{\omega\mp\omega_c+i/\tau}\right) = \omega^2\epsilon(\omega),$$

where

$$\epsilon(\omega) = 1-\frac{\omega_p^2}{\omega}\frac{1}{\omega\mp\omega_c+i/\tau},$$
$$\omega_p^2 = \frac{4\pi ne^2}{m}.$$

(c) For polarization $E_y=iE_x$,

$$\epsilon(\omega) = 1-\frac{\omega_p^2}{\omega}\frac{1}{\omega-\omega_c+i/\tau}.$$

(SKETCH/PLOT?...)

Assuming $\omega_p/\omega_c \gg 1$ and $\omega_c\tau\gg 1$, for large ω, one can rewrite the above eq. as

$$\epsilon(\omega)=1-\frac{\omega_p}{\omega}\frac{1}{\frac{\omega}{\omega_p}-\frac{\omega_c}{\omega_p}+\frac{i}{\tau\omega_p}}\approx 1-\frac{\omega_p}{\omega}\frac{1}{\frac{\omega}{\omega_p}} = 1-\frac{\omega_p^2}{\omega^2},$$

which is positive for $\omega>\omega_p$, and real solutions for k exist.

For small but positive ω, one has

$$\epsilon(\omega)=1+\frac{\omega_p^2}{\omega}\frac{1}{\omega_c-\omega-i/\tau},$$

which, if τ is larger and therefore the i/τ term is ignored, is positive for $\omega<\omega_c$, and consequently real solutions for k exist.

(d) For $\omega\ll\omega_c$ (but still >0),

$$\epsilon(\omega) \approx 1-\frac{\omega_p^2}{\omega}\frac{1}{-\omega_c}\approx\frac{\omega_p^2}{\omega\omega_c},$$
$$k^2c^2 = \epsilon\omega^2\approx\frac{\omega_p^2}{\omega\omega_c}\omega^2=\frac{\omega_p^2}{\omega_c}\omega,$$
$$\omega = \omega_c\frac{k^2c^2}{\omega_p^2}.$$

$\lambda=1$ cm, $T=10$ kilogauss. $c=3\times10^{10}$ cm/s, $e=4.8\times10^{-10}$ esu. Taking a typical metalic electron density of $10^{23}/\text{cm}^3$, the helicon frequency is

$$f=\frac{\omega}{2\pi}=\frac{eH}{mc}\frac{k^2c^2}{\frac{4\pi ne^2}{m}}\frac{1}{2\pi}=\frac{Hc}{8\pi^2ne}k^2=\frac{Hc}{8\pi^2ne}\left(\frac{2\pi}{\lambda}\right)^2=\frac{Hc}{2ne}\frac{1}{\lambda^2}=\frac{(10^4)(3\times10^{10})}{(2)(10^{23})(4.8\times10^{-10})}\frac{1}{(1)^2}=3.1\text{ Hz}.$$