# Homework 6, due December 6, 1996

Problem 1

Part a)
At $x \leq -\frac{a}{2}$ $\psi(x) = Ae^{iKx} + (Ar + Bt)e^{-iKx}$
At $x \geq \frac{a}{2}$ $\psi(x) = (At + Br)e^{iKx} + Be^{-iKx}$
From (8.68) $\psi(\frac{a}{2}) = e^{ika}\psi(-\frac{a}{2}) = e^{ika}(Ae^{-iK\frac{a}{2}} + (Ar + Bt)e^{iK\frac{a}{2}})$
which is equal to $(At + Br)e^{iK\frac{a}{2}} + Be^{-iK\frac{a}{2}}$
which gives
$e^{ika}(A + (Ar + Bt)e^{iKa}) = B + (At + Br)e^{iKa}$
or
$(e^{ika}(1 + re^{iKa}) - te^{iKa})A + (e^{ika}te^{iKa} - 1 - re^{iKa})B = 0$
Similarly from the derivative condition:
$e^{ika}(AK - (Ar + Bt)Ke^{iKa}) = -BK + (At + Br)Ke^{iKa}$
leads to
$(e^{ika}(1 - re^{iKa}) - te^{iKa})A + (-e^{ika}te^{iKa} + 1 - re^{iKa})B = 0$
Set the determinant equal to zero:
$(e^{ika}(1 + re^{iKa}) - te^{iKa})(-e^{ika}te^{iKa} + 1 - re^{iKa}) -$
$(e^{ika}te^{iKa} - 1 - re^{iKa})(e^{ika}(1 - re^{iKa}) - te^{iKa}) = 0$
Combine terms in the exponent of big K:
$e^{2iKa}(t^2 - r^2)2e^{ika} - 2t(1 + e^{i2ka})e^{iKa} + 2e^{ika} = 0$
Multiply by $e^{-ika}e^{-iKa}$ to get
$e^{iKa}(t^2 - r^2) - t(e^{-ika} + e^{ika}) + e^{-iKa} = 0$
and combine exponents in a cosine:
$2t\cos(ka) = e^{iKa}(t^2 - r^2) + e^{-iKa}$
For v=0 we have r=0, t=1, and indeed we get k=K.
Part b)
$\frac{dw}{dx} = \frac{d^2\phi_1}{dx^2}\phi_2 - \phi_1\frac{d^2\phi_2}{dx^2}$
use Schrödinger's equation to show
$\frac{\hbar^2}{2m}\frac{dw}{dx} = (v - E)\phi_1\phi_2 - \phi_1(v - E)\phi_2 = 0$
Part c)
$w(\phi_l, \phi_l^*) = |t|^2 2iK$ for $x \geq \frac{a}{2}$ and $w(\phi_l, \phi_l^*) = 2iK(1 - |r|^2)$ for $x \leq -\frac{a}{2}$. These have to be the same, hence (8.72) is correct.
Part d)
$w(\phi_l, \phi_r^*) = 2iKtr^*$ for $x \geq \frac{a}{2}$ and also $w(\phi_l, \phi_r^*) = -2iKt^*r$ for $x \leq -\frac{a}{2}$
hence $tr^* = -t^*r$ and $rt^*$ is purely imaginary. This gives (8.75)
Part e)
$2t\cos(ka) = e^{iKa}(t^2 - r^2) + e^{-iKa}$ becomes
$2|t|e^{i\delta}\cos(ka) = e^{iKa}(|t|^2e^{i2\delta} + |r|^2e^{i2\delta}) + e^{-iKa}$
or using (8.72)
$2|t|e^{i\delta}\cos(ka) = e^{iKa}e^{i2\delta} + e^{-iKa}$
multiply by $e^{-i\delta}$ to get
$2|t|\cos(ka) = e^{i(Ka+\delta)} + e^{-i(Ka+\delta)}$

and we have (8.76)

Part f)

Work in the gap containing $Ka = n\pi - \delta$ and write $Ka = n\pi - \delta + \Delta Ka$ Solve for $\cos(Ka + \delta) = \pm|t|$

This gives $\cos(n\pi + \Delta Ka) = \pm|t|$ The sign is determined by the value of n and we get for small values:

$1 - \frac{1}{2}(\Delta Ka)^2 \approx |t|$ or $\Delta Ka \approx \sqrt{2(1-|t|)}$

Now $|r|^2 = 1 - |t|^2 = (1-|t|)(1+|t|) \approx 2(1-|t|)$

and hence $\Delta Ka \approx |r|$

The energy gap is

$\epsilon_{gap} \approx \frac{\hbar^2}{2m} 2K 2\Delta K = \frac{\hbar^2}{ma^2} 2Ka|r|$ which gives (8.78). Note the factor of two because we have to go from minus to plus $\Delta K$.

Part g)

If $|t|$ is very small we see that $\cos(Ka + \delta) = \pm|t|$ tells us that Ka is close to half integral values of pi. Write $Ka = n\pi + \frac{\pi}{2} - \delta + \Delta Ka$ to get $\sin(n\pi + \Delta Ka) = \pm|t|$ Again the sign follows from the value of n. This gives $\Delta Ka \approx |t|$ and now the band width is very narrow and of order $|t|$.

Part h)

A delta function potential. We have

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + g\delta(x)\psi = E\psi$

This is a standard example. Use $\psi_l$ to get

$x < 0$ $\psi(x) = Ae^{iKx} + Are^{-iKx}$

$x > 0$ $\psi(x) = Ate^{iKx}$

The wave function is continuous, hence $1+r = t$. By integrating Schrödinger's equation we find

$-\frac{\hbar^2}{2m}(\frac{d\psi}{dx}(0^+) - \frac{d\psi}{dx}(0^-)) + g\psi(0) = 0$

or

$(\frac{d\psi}{dx}(0^+) - \frac{d\psi}{dx}(0^-)) = \frac{2mg}{\hbar^2}\psi(0)$

This gives

$iKt - (iK - iKr) = \frac{2mg}{\hbar^2}t$ or $t + r - 1 = -i\frac{2mg}{\hbar^2 K}t$

Substitute r to get

$t(1 + i\frac{mg}{\hbar^2 K}) = 1$

or

$t = \frac{1}{1+i\frac{mg}{\hbar^2 K}}$

This gives for the phase

$\cot(\delta) = \frac{real(t)}{imaginary(t)} = -\frac{\hbar^2 K}{mg}$

and for the norm:

$|t|^2 = \frac{1}{1+(\frac{mg}{\hbar^2 K})^2} = \frac{1}{1+\tan^2(\delta)} = \cos^2(\delta)$