Solutions for Homework Set 7

Jock McOrist

Note: In these solutions I've used only SI units, and the problem numbers here refer to the 7th edition of *Kittel*.

1. *Kittel Problem 9.11*

Solution

(a) We are given the hamiltonian for the electron to be

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right) + \frac{1}{2m}\left[-i\hbar\frac{\partial}{\partial x} - \frac{eyB}{c}\right]^2,$$

and substituting the ansatz $\psi = \chi(y)\exp[i(k_x x + k_z z)]$ in the wave equation $H\psi = E\psi$ gives

$$\begin{aligned} H\psi &= -\frac{\hbar^2\chi(y)''}{2m}e^{i(k_x x+k_z z)} + \left(\frac{\hbar^2(k_z^2+k_x^2)}{2m} - \frac{\hbar eBk_x}{mc}y + \frac{e^2B^2}{2mc}y^2\right)\chi(y)e^{i(k_x x+k_z z)}, \\ &= -\frac{\hbar^2\chi(y)''}{2m}e^{i(k_x x+k_z z)} + \left(\frac{\hbar^2k_z^2}{2m} + \frac{1}{2}m\omega_c^2(y-y_0)^2\right)\chi(y)e^{i(k_x x+k_z z)}, \end{aligned} \tag{1}$$

where we've factorised the last three terms in the last line using $y_0 = c\hbar k_x/eB$ and $\omega_c = eB/mc$. Equating this to $\epsilon\psi$ gives us an equation for $\chi(y)$:

$$(\epsilon - H)\psi = \frac{\hbar^2\chi(y)''}{2m} + \left(\epsilon - \frac{\hbar^2k_z^2}{2m} - \frac{1}{2}m\omega_c^2(y-y_0)^2\right)\chi(y) = 0. \tag{2}$$

(b) Taking (2), we let $E = \epsilon - \hbar^2k_z^2/2m$ and make the change of variables $\tilde{y} = y - y_0$ we get

$$\frac{-\hbar^2\chi(\tilde{y})''}{2m} + \frac{1}{2}m\omega_c^2\tilde{y}^2\chi(y) = E\chi(\tilde{y}) \tag{3}$$

But this is exactly the Schroedinger wave equation for a simple harmonic oscillator with frequency ω_c and energy E. Hence, the quantised energy levels are given by $E_n = (n+1/2)\hbar\omega$, and so

$$\epsilon_n = \frac{\hbar^2k_z^2}{2m} + (n+\frac{1}{2})\hbar\omega,$$

are the energy levels of the harmonic oscillator for $\chi(y)$.

2. *Kittel Problem 12.1*

Solution

(a) The penetration equation we need to solve is a wave equation for B inside a superconducting plate. By symmetry the equation reduces to a one dimensional problem in x. By inspection the penetration equation

$$\lambda^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}B = B,$$

has expotential solutions. Hence, the most general solution we can write down is given by

$$B(x) = A\cosh(x/\lambda) + B\sinh(x/\lambda), \tag{4}$$

where A and B are constants to be determined by the boundary conditions. We are given that the magnetic field outside the plate is constant and parallel to the plate, and so this gives us two boundary conditions on B:

$$B(\delta/2) = B(-\delta/2) = B_a. \tag{5}$$

Applying these two the solution (4) gives:

$$\begin{aligned} B(\delta/2) &= A\cosh(\delta/2\lambda) + B\sinh(\delta/2\lambda) = B_a, \\ B(-\delta/2) &= A\cosh(\delta/2\lambda) - B\sinh(\delta/2\lambda) = B_a. \end{aligned} \tag{6}$$

From this, solving for A and B gives $B = 0$, and

$$A = \frac{B_a}{\cosh(\delta/2\lambda}.$$

Hence,

$$B(x) = B_a \frac{\cosh(x/\lambda)}{\cosh(\delta/2\lambda}. \tag{7}$$

(b) The effective magnetisation is defined by

$$B(x) - B_a = \mu_0 M(x).$$

Using (7) gives

$$B_a \left[\frac{\cosh(x/2\lambda) - \cosh(\delta/2\lambda)}{\cosh(\delta/2\lambda)}\right] = \mu_0 M(x). \tag{8}$$

For a thin plate we have $\delta \ll \lambda$, and so for $x \ll 1$ we use the expansion $\cosh x \approx 1 + x^2/2$ to give

$$\begin{aligned}
\mu_0 M(x) &= B_a \frac{x^2/8\lambda^2 - \delta^2/8\lambda^2}{1 + (\delta/2\lambda)^2}, \\
&= -\frac{B_a}{2}\left[\frac{\delta^2}{4\lambda^2} - \frac{x^2}{4\lambda^2}\right]. \qquad (9)
\end{aligned}$$

3. *Kittel Problem 12.3*

Solution

(a) We make use of the two Maxwell equations:

$$\begin{aligned}
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}, \qquad (10) \\
\nabla \times \mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B}. \qquad (11)
\end{aligned}$$

We split up the source current into a normal and superconducting contribution $\mathbf{J} = \mathbf{J}_N + \mathbf{J}_S$. The superconducting contribution obeys the London equation, taking the curl of which gives

$$\nabla \times \mathbf{J}_S = -\frac{1}{\lambda^2 \mu_0}\mathbf{B}, \tag{12}$$

while the normal contribution obeys Ohm's law

$$\mathbf{J}_N = \sigma_0 \mathbf{E},$$

where σ_0 is the conductivity. Taking the curl of (10) gives

$$\nabla^2 \mathbf{B} = -\mu_0 \sigma_0 \nabla \times \mathbf{E} + \mu_0 \left(\frac{1}{\lambda^2 \mu_0}\right)\mathbf{B} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}, \tag{13}$$

where we used (11), (12) and $\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B}$. Now using the second Maxwell equation (11) gives

$$\nabla^2 \mathbf{B} = \mu_0 \sigma_0 \frac{\partial}{\partial t}\mathbf{B} + \frac{1}{\lambda^2}\mathbf{B} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}, \tag{14}$$

We now look for wavelike solutions substituting $\mathbf{B} = \mathbf{B}_0 \exp[i(kr - \omega)]$ into (14). We then get the dispersion equation for wavelike solutions:

$$\begin{aligned}
-k^2 &= \frac{1}{\lambda^2} + \mu_0 \sigma_0 (-i\omega) + \frac{\omega^2}{c^2}, \qquad (15) \\
c^2 k^2 &= -\frac{c^2}{\lambda^2} + i\frac{\sigma_0}{\epsilon_0}\omega + \omega^2 \qquad (16)
\end{aligned}$$

(b) The conductivity for normal electrons is $\sigma_0 = n_N e^2 \tau / m$ with τ the relaxation time and so assuming $\omega\tau \ll 1$ we have

$$k^2 c^2 = -\frac{c^2}{\lambda^2} + \omega^2, \tag{17}$$

and rearranging for ω^2 gives:

$$\omega^2 = \left(\frac{c^2}{\lambda^2} + k^2 c^2\right). \tag{18}$$

Therefore the motion of normal electrons is not important in this limit.

4. *Kittel Problem 12.3*

Solution

(a) The London equation is given by

$$\mathbf{J} = -\frac{c}{\mu_0 \lambda^2}\mathbf{A}. \tag{19}$$

Taking the time deriviative gives:

$$\frac{\partial}{\partial t}\mathbf{J} = \frac{\mathbf{E}}{\mu_0 \lambda^2}, \tag{20}$$

where we use the relation

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}. \tag{21}$$

(b) Suppose

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{E},$$

and since $\mathbf{J} = nq\mathbf{v}$ then

$$\begin{aligned}\left(\frac{m}{ne}\right)\frac{\partial}{\partial t}\mathbf{J} &= q\mathbf{E} \\ \frac{m\mathbf{E}}{ne\mu_0\lambda^2} &= q\mathbf{E}. \end{aligned} \tag{22}$$

Hence,

$$\lambda^2 = \frac{m}{neq\mu_0}. \tag{23}$$