Homework 5, due March 10, 1999

Problem 1. Ashcroft-Mermin 13.6

(a) Start with (12.33)

$\hbar \frac{d}{dt}\vec{k} = -\frac{eH}{c}\vec{v}(\vec{k}) \times \hat{H}$

now use

$< \vec{v}(\vec{k}) >= \int_{-\infty}^{0} \frac{dt}{\tau(\vec{k})} e^{\frac{t}{\tau(\vec{k})}} \vec{v}(\vec{k}(t))$

where we supressed the bandindex and where we consider a trajectory with $\vec{k}(t=0) = \vec{k}$. I also changed notation for the average for typographical reasons. Integration of (12.33) gives

$-\frac{c\hbar}{eH} \int_{-\infty}^{0} \frac{dt}{\tau(\vec{k})} e^{\frac{t}{\tau(\vec{k})}} \frac{d}{dt}\vec{k}(t) = \int_{-\infty}^{0} \frac{dt}{\tau(\vec{k})} e^{\frac{t}{\tau(\vec{k})}} \vec{v}(\vec{k}(t)) \times \hat{H} =< \vec{v}(\vec{k}) > \times \hat{H}$

This tells us that the change in $\vec{k}$ is perpendicular to the field and hence we can also write

$< \vec{v}(\vec{k}) >_{\perp}= -\frac{c\hbar}{eH}\hat{H} \times \int_{-\infty}^{0} \frac{dt}{\tau(\vec{k})} e^{\frac{t}{\tau(\vec{k})}} \frac{d}{dt}\vec{k}_{\perp}(t)$

The relaxation time only depends on the energy and that is conserved, hence we leave it outside and get

$< \vec{v}(\vec{k}) >_{\perp}= -\frac{c\hbar}{eH\tau}\hat{H} \times \int_{-\infty}^{0} dt e^{\frac{t}{\tau}} \frac{d}{dt}\vec{k}_{\perp}(t)$

Integration by parts gives

$< \vec{v}(\vec{k}) >_{\perp}= -\frac{c\hbar}{eH\tau}\hat{H} \times \left(\vec{k}_{\perp} - \int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_{\perp}(t) \right)$

For a closed orbit we have

$\int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_{\perp}(t) = \sum_{n=0}^{\infty} \int_{-(n+1)T}^{nT} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_{\perp}(t) = \sum_{n=0}^{\infty} e^{-\frac{nT}{\tau}} \int_{-T}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_{\perp}(t)$

Now we use

$\sum_{n=0}^{\infty} e^{-\frac{nT}{\tau}} = \frac{1}{1-e^{-\frac{T}{\tau}}}$

In the high field limit we have $T << \tau$ and hence

$\sum_{n=0}^{\infty} e^{-\frac{nT}{\tau}} \approx \frac{\tau}{T}$

with corrections of order $\frac{1}{H}$. This gives

$\int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_{\perp}(t) = \frac{1}{T} \int_{-T}^{0} dt e^{\frac{t}{\tau}} \vec{k}_{\perp}(t)$

to order $\frac{1}{H^2}$ In addition, we also have

$$\frac{1}{T}\int_{-T}^{0} dt e^{\frac{t}{\tau}} \vec{k}_\perp(t) \approx \frac{1}{T}\int_{-T}^{0} dt \vec{k}_\perp(t)$$

for the same reason and to the same order. This proves (13.85). NOTE: symmetry actually makes the last integral equal to zero, unless spin-orbit coupling is included.!

(b) The derivation is the same to

$$\int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_\perp(t) = \sum_{n=0}^{\infty} \int_{-(n+1)T}^{nT} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_\perp(t)$$

but now the time T is the time it takes to go through one Brillouin zone on the open orbit. The distance travelled in this time T is therefore some reciprocal lattice vector $\vec{K}$. In this case we have

$$\vec{k}(t+T) = \vec{k}(t) + \vec{K}$$

and we get

$$\int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_\perp(t) = \sum_{n=0}^{\infty} \int_{-T}^{0} e^{-\frac{nT}{\tau}} dt \frac{1}{\tau} e^{\frac{t}{\tau}} (\vec{k}_\perp(t) + n\vec{K}_\perp)$$

With

$$\sum_{n=0}^{\infty} e^{-\frac{nT}{\tau}} = \frac{1}{1-e^{-\frac{T}{\tau}}} \approx \frac{\tau}{T}$$

and

$$\sum_{n=0}^{\infty} n e^{-\frac{nT}{\tau}} = -\tau \frac{d}{dT} \sum_{n=0}^{\infty} e^{-\frac{nT}{\tau}} \approx -\tau \frac{d}{dT}\frac{\tau}{T} = \frac{\tau^2}{T^2}$$

this gives

$$\int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_\perp(t) \approx \int_{-T}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} (\frac{\tau}{T}\vec{k}_\perp(t) + \frac{\tau^2}{T^2}\vec{K}_\perp)$$

and the second term is much larger, hence

$$\int_{-\infty}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \vec{k}_\perp(t) \approx \int_{-T}^{0} dt \frac{1}{\tau} e^{\frac{t}{\tau}} \frac{\tau^2}{T^2}\vec{K}_\perp = \frac{\tau}{T}\vec{K}_\perp$$

and we get

$$< \vec{v}(\vec{k}) >_\perp = -\frac{c\hbar}{eH\tau}\hat{H} \times \left(\vec{k}_\perp - \frac{\tau}{T}\vec{K}_\perp\right) \approx \frac{c\hbar}{eHT}\hat{H} \times \vec{K}_\perp = \frac{\hbar}{2\pi m^*}\hat{H} \times \vec{K}_\perp$$

The expression

$$\frac{1}{T}\vec{K}_\perp$$

is nothing but the average velocity along the orbit in K-space and therefore $< \vec{v}(\vec{k}) >_\perp$ is just the average velocity of motion along the orbit in real space.

(c) We use (13.69) for the conductivity

$$\tilde{\sigma} \cdot \vec{E} = e^2 \int \frac{d^3k}{4\pi^3} \tau(\epsilon(\vec{k}))\vec{v}(\vec{k}) < \vec{v}(\vec{k}) \cdot \vec{E} > \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{k}))}$$

Because the derivative of the Fermi function is a delta function we use ony the relaxation time at the Fermi energy, τ. Also we use (13.26) to change the derivative in a k-derivative and get

$$\vec{j} = \frac{e^2}{\hbar}\tau \int \frac{d^3k}{4\pi^3} < \vec{v}(\vec{k}) \cdot \vec{E} > \left(-\frac{\partial f}{\partial \vec{k}}\right)$$

Now we take $\vec{E} \cdot \vec{H} = 0$, which means that the component of $< \vec{v}(\vec{k}) >$ parallel to $\hat{H}$ does not contribute to the dotproduct and we get

$$\vec{j} = \frac{e^2}{\hbar}\tau \int \frac{d^3k}{4\pi^3} < \vec{v}_\perp(\vec{k}) \cdot \vec{E} > \left(-\frac{\partial f}{\partial \vec{k}}\right)$$

Therefore, for a closed orbit,

$$\vec{j} = \frac{e^2}{\hbar}\tau \int \frac{d^3k}{4\pi^3} \frac{-\hbar c}{eH\tau} \hat{H} \times (\vec{k} - < \vec{k} >)_\perp \cdot \vec{E} \left(-\frac{\partial f}{\partial \vec{k}}\right)$$

because in the cross product we only need the perpendicular component of the k-vector. With $\vec{w} = \frac{c}{H}\vec{E} \times \hat{H}$ we get

$$\vec{j} = -e \int \frac{d^3k}{4\pi^3} \vec{w} \cdot (\vec{k} - < \vec{k} >)_\perp \left(-\frac{\partial f}{\partial \vec{k}}\right)$$

The term with $< \vec{k} >$ does not give a contribution, since we can do an integration over $\vec{k}_\perp$ first and $< \vec{k} >$ does not depend on this (already integrated out) and since f is periodic the integral is zero. Therefore

$$\vec{j} = -e \int \frac{d^3k}{4\pi^3} \vec{w} \cdot \vec{k}_\perp \left(-\frac{\partial f}{\partial \vec{k}}\right)$$

Since we also have $\vec{w} \cdot \vec{k}_{||} = 0$ this is equal to (13.87) for the complete $\vec{j}$.

Consider a closed electron Fermi surface. Take the origin of the Brillouin zone at the center of this orbit. This choice is allowed. We then have

$$\vec{j} = -e \int \frac{d^3k}{4\pi^3} \frac{\partial \vec{w} \cdot \vec{k}}{\partial \vec{k}} f + e \int_{surface} \frac{d^2\vec{S}}{4\pi^3} \vec{w} \cdot \vec{k} f$$

But if the orbit is a closed electron orbit, the Fermi function is zero outside and the surface integral vanishes. Hence

$$\vec{j} = -e \int \frac{d^3k}{4\pi^3} \frac{\partial \vec{w} \cdot \vec{k}}{\partial \vec{k}} f = -e \int \frac{d^3k}{4\pi^3} \vec{w} f = -n_e e \vec{w}$$

where n_e is the number of electrons inside the electron surface. Since $\vec{w}$ is perpendicular to $\hat{H}$, so is $\vec{j}$. For a hole surface we rewrite the original formula slightly:

$$\vec{j} = -e \int \frac{d^3k}{4\pi^3} \vec{w} \cdot \vec{k} \left(\frac{\partial(1-f)}{\partial \vec{k}}\right)$$

Integration by parts gives

$$\vec{j} = e \int \frac{d^3k}{4\pi^3} \frac{\partial \vec{w} \cdot \vec{k}}{\partial \vec{k}} (1-f) - e \int_{surface} \frac{d^2\vec{S}}{4\pi^3} \vec{w} \cdot \vec{k} (1-f)$$

the surface term disappears again, since for a hole Fermi surface the Fermi function is one outside. We now have

$\vec{j} = e \int \frac{d^3k}{4\pi^3} \vec{w}(1-f) = n_h e \vec{w}$

as required.

(d) Now we describe a Fermi surface that is open. In this case we have

$\vec{j} = e^2 \tau \int \frac{d^3k}{4\pi^3} \vec{v}(\vec{k}) < \vec{v}(\vec{k}) \cdot \vec{E} > \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{k}))}$

The direction of the orbit in real space $\hat{n}$ is parallel to $< \vec{v}(\vec{k}) >$ and hence at high magnetic fields we have a part $\hat{n} \cdot \vec{E}$ just like in (12.56). Note that the second term in (12.56) vanishes at high fields.

Now start at a point $\vec{k}$ and perform a line integral along $\vec{K}_\perp$ as part of the previous expression for $\vec{j}$. Since the average velocity is the same along this orbit we need to integrate the first velocity term along this line. Since the integral is periodic, only terms perpendicular to $\vec{K}_\perp$ survive (the average of a parallel term along this line is zero). Since we also know that the current is perpendicular to $\hat{H}$, the current will be parallel to $\hat{n}$

In part (b) we showed that the average velocity at high fields is independent of H, and hence the current is independent of H.

(e) The conductivity formula is

$\tilde{\sigma} = e^2 \tau \int \frac{d^3k}{4\pi^3} \vec{v}(\vec{k}) < \vec{v}(\vec{k}) > \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{k}))}$

This has an explicit τ dependence in front, and the rest only comes via

$< \vec{v}(\vec{k}) >= \int\limits_{-\infty}^{0} \frac{dt}{\tau(\vec{k})} e^{\frac{t}{\tau(\vec{k})}} \vec{v}(\vec{k}(t)) = \int\limits_{-\infty}^{0} dt' e^{t'} \vec{v}(\vec{k}(t'\tau))$

We have

$\frac{d}{dt'} \vec{v}(\vec{k}(t'\tau)) = \frac{\partial \vec{v}}{\partial \vec{k}} \cdot \frac{d\vec{k}}{dt} \tau = \pm \hbar \tau \tilde{M}^{-1} \frac{d\vec{k}}{dt}$

The conductivity is defined in the limit $\vec{E} \rightarrow 0$ and hence we use

$\frac{d}{dt'} \vec{v}(\vec{k}(t'\tau)) = \pm \frac{-e}{c} \hbar \tau \tilde{M}^{-1} \vec{v} \times \vec{H}$

and this only depends on the combination $H\tau$. Therefore also

$\tilde{\rho} = \tau \tilde{G}(H\tau)$

and Kohler's law follows immediately:

$\frac{\rho_{xx}(H) - \rho_{xx}(0)}{\rho_{xx}(0)} = \frac{\tau G_{xx}(H\tau) - \tau G_{xx}(0)}{\tau G_{xx}(0)} = \frac{G_{xx}(H\tau) - G_{xx}(0)}{G_{xx}(0)}$

(f) Starting from

$\tilde{\sigma} = e^2 \tau \int \frac{d^3k}{4\pi^3} \vec{v}(\vec{k}) < \vec{v}(\vec{k}) > \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{k}))}$

and using the definition of the average we have

$$\tilde{\sigma} = e^2\tau \int \frac{d^3k}{4\pi^3}\vec{v}(\vec{k}) \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{k}))} \int\limits_{-\infty}^{0} \frac{dt}{\tau(\vec{k})} e^{\frac{t}{\tau(\vec{k})}} \vec{v}(\vec{k}(t))$$

Again using the fact that we only need the lifetime at the Fermi level we rewrite after interchanging the integrals:

$$\tilde{\sigma} = e^2 \int\limits_{-\infty}^{0} dt e^{\frac{t}{\tau}} \int \frac{d^3k}{4\pi^3}\vec{v}(\vec{k}) \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{k}))} \vec{v}(\vec{k}(t))$$

Now replace $\vec{k}(t) = \vec{q}$ and $\vec{k} = \vec{q}(-t)$ and use Liouville's theorem telling that phase space volumes are the same

$$\tilde{\sigma}(\vec{H}) = e^2 \int\limits_{-\infty}^{0} dt e^{\frac{t}{\tau}} \int \frac{d^3q}{4\pi^3}\vec{v}(\vec{q}(-t)) \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{q}))} \vec{v}(\vec{q})$$

where also we used the fact that the energy is conserved in magnetic field. In this expression we need $\vec{q}$ at a later time. But from the equation of motion we see that we traverse the orbit in opposite direction if we invert the magnetic field. Therefore

$$\tilde{\sigma}(-\vec{H}) = e^2 \int\limits_{-\infty}^{0} dt e^{\frac{t}{\tau}} \int \frac{d^3q}{4\pi^3}\vec{v}(\vec{q}(+t)) \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{q}))} \vec{v}(\vec{q}) =$$

$$e^2\tau \int \frac{d^3q}{4\pi^3} < \vec{v}(\vec{q}) > \left(-\frac{\partial f}{\partial \epsilon}\right)_{(\epsilon=\epsilon(\vec{q}))} \vec{v}(\vec{q})$$

which has the velocities in opposite order. This then shows

$$\sigma_{\mu\nu}(\vec{H}) = \sigma_{\nu\mu}(-\vec{H})$$

as required.