Solutions for Homework Set 5

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1. *Kittel Problem 6.4*

Solution

(a) The mass of the sun is $M = 2 \times 10^{33}$g, which we make a number of crude assumptions:

(i) The sun is composed entirely of H and it is completely ionised of its electrons. Also, the number densities of electrons and protons are of equal number: $N_p = N_e$,

(ii) the mass of the electrons are negligible to that of the protons i.e. $N_e m_e \ll N_p m_p$,

(iii) we can treat the Sun and white dwarf as a free electron gas, with N_e electrons.

Hence, using (i) and (ii) we have

$$\begin{aligned} N_e &= \frac{M}{m_p}, \\ &\approx 1.20 \times 10^{57}. \end{aligned} \quad (1)$$

The Fermi-energy is given by

$$\varepsilon_f = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N_e}{V}\right)^{2/3}. \quad (2)$$

Inside a white dwarf of radius $R = 2 \times 10^7$m, the volume is $V = 4\pi R^3/3 = 3.35 \times 10^{22}\text{m}^3$ and using this together with (1) and (2) we have

$$\epsilon_f = 40 \text{ keV}. \quad (3)$$

(b) In the ultra-relativistic limit, the rest mass of the electron is negligible, and consequently the dispersion equation becomes $\epsilon = \hbar kc$. The fermi-wavevector is given by

$$k_f = \left(\frac{3\pi^2 N_e}{V}\right)^{1/3}. \quad (4)$$

Hence,

$$\epsilon_f = \left(\frac{3\pi^2 N}{V}\right)^{1/3} \hbar c \approx \hbar c (N/V)^{1/3}. \quad (5)$$

(c) If the radius $R = 10$km, the volume of the sphere is $V = 4.2 \times 10^{12}\text{m}^3$. If one were to calculate the Fermi-energy using the non-relativistic expression then one would get a value of $\epsilon_f = 1.5 \times 10^5\text{MeV} \gg 0.511\text{MeV} = mc^2$, which is relativistic itself! Hence, one should use the relativistic expression in (b) above. Doing this gives

$$\epsilon_f \approx 1.3 \times 10^2\text{MeV}. \quad (6)$$

2. *Kittel Problem 6.6*

Solution

Given the equation

$$m\left(\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{v(t)}{\tau}\right) = -eE(t),$$

we can analyse the Fourier component by substituting $E(t) = E(\omega)\exp(-i\omega t)$, and $v(t) = v(\omega)\exp(-i\omega t)$. Doing this and solving for $v(\omega)$ gives

$$v(\omega) = \frac{-em/\tau}{1 - i\omega\tau} E(\omega). \quad (7)$$

The current density is given by $j = -nev$, and hence using $\sigma_0 = ne^2\tau/m$ we have

$$\sigma(\omega) = \sigma_0 \frac{1 + i\omega\tau}{1 + \omega^2\tau^2}, \quad (8)$$

where we use the definition of the conductivity given by

$$j(\omega) = \sigma(\omega) v(\omega).$$

3. *Kittel Problem 6.7*

Solution

(a) With a magnetic field one must now include the complete Lorentz force, and the appropiate equations of motion to use are given in Kittel (6.51). Taking the Fourier components of these equations gives

$$\begin{aligned} m(-i\omega v_x + \frac{v_x}{\tau}) &= -e(E_x + \frac{Bv_y}{c}), \qquad (9) \\ m(-i\omega v_y + \frac{v_y}{\tau}) &= -e(E_y - \frac{Bv_x}{c}). \qquad (10) \end{aligned}$$

Rewriting this as a matrix equation using the definition of the cyclotron frequency viz. $\omega_c = eB/mc$,

$$\begin{pmatrix} 1-i\omega\tau & \omega_c\tau \\ -\omega_c\tau & 1-i\omega\tau \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = -\frac{e\tau}{m} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \qquad (11)$$

Inverting this matrix equation for (v_x, v_y) one has

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -\frac{e\tau/m}{(1-i\omega\tau)^2 + (\omega_c\tau)^2} \begin{pmatrix} 1-i\omega\tau & -\omega_c\tau \\ \omega_c\tau & 1-i\omega\tau \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \qquad (12)$$

Hence, using $\mathbf{j} = -ne\mathbf{v}$ gives

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \frac{ne^2\tau/m}{(1-i\omega\tau)^2 + (\omega_c\tau)^2} \begin{pmatrix} 1-i\omega\tau & -\omega_c\tau \\ \omega_c\tau & 1-i\omega\tau \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \qquad (13)$$

Introducing the plasma fequency defined by $\omega_p^2 = 4\pi ne^2/m$ gives

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \frac{\omega_p^2\tau/4\pi}{(1-i\omega\tau)^2 + (\omega_c\tau)^2} \begin{pmatrix} 1-i\omega\tau & -\omega_c\tau \\ \omega_c\tau & 1-i\omega\tau \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \qquad (14)$$

We now make the high frequency assumptions that $\omega\tau \gg 1$ and $\omega \gg \omega_c$ which simplifies our expression. Doing this, with some algebra gives

$$\begin{aligned} \begin{pmatrix} j_x \\ j_y \end{pmatrix} &= \frac{\omega_p^2\tau}{4\pi(\omega\tau)^2} \begin{pmatrix} i\omega\tau & \omega_c\tau \\ -\omega_c\tau & i\omega\tau \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \\ &= \frac{\omega_p^2}{4\pi\omega^2} \begin{pmatrix} i\omega & \omega_c \\ -\omega_c & i\omega \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \qquad (15) \end{aligned}$$

Thus, using the definition that $j_i = \sigma_{ij}E_j$, we have for the conductivity tensor:

$$\sigma_{xx} = \sigma_{yy} = i\frac{\omega_p^2}{4\pi\omega}, \qquad (16)$$

and

$$\sigma_{xy} = -\sigma_{yx} = \frac{\omega_p^2\omega_c}{4\pi\omega^2}. \qquad (17)$$

(b) The dielectric tensor is given by

$$\epsilon_{ij} = \delta_{ij} + \frac{4\pi i\sigma_{ij}}{\omega},$$

while the wave equation for the electric field is given by

$$\nabla^2\mathbf{E} = \epsilon\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Hence, for a plane wave ansatz $\mathbf{E} = \mathbf{E_0}\exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t))$, with $\mathbf{k} = k\hat{\mathbf{z}}$ one has

$$\begin{aligned} c^2k^2E_x &= \omega^2((1 + \frac{4\pi i}{\omega}\sigma_{xx})E_x + \frac{4\pi i}{\omega}\sigma_{xy}E_y), \qquad (18) \\ c^2k^2E_y &= \omega^2(\frac{4\pi i}{\omega}\sigma_{yx}E_x + (1 + \frac{4\pi i}{\omega}\sigma_{yy})E_y). \qquad (19) \end{aligned}$$

Rewriting as a matrix equation

$$\begin{pmatrix} c^2k^2-\omega^2+\omega_p^2 & -i\omega_p^2\,\omega_c/\omega \\ i\omega_p^2\,\omega_c/\omega & c^2k^2-\omega^2+\omega_p^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

For a solution to exist the determinant must vanish. This gives us a secular equation which is the dispersion equation. Explicitily,

$$(c^2k^2-\omega^2+\omega_p^2)^2-(\omega_p^2\,\omega_c/\omega)^2 = 0. \tag{21}$$

Solving for c^2k^2 gives

$$c^2k^2=\omega^2-\omega_p^2\pm\frac{\omega_p^2\,\omega_c}{\omega}. \tag{22}$$

4. *Kittel Problem 6.9*

Solution

The solution to this problem goes along the lines of the solution to problem *6.7* above, except now for a static solution the time derivatives all vanish. Therefore, equation (51) in *Kittel* becomes

$$\begin{aligned} \frac{m}{\tau}v_x+(eB/c)v_y &= -eE_x; \\ \frac{m}{\tau}v_y-(eB/c)v_x &= -eE_y; \\ \frac{m}{\tau}v_z &= -eE_z. \end{aligned} \tag{23}$$

As before we write a matrix equation as

$$\begin{pmatrix} 1 & \omega_c\tau & 0 \\ -\omega_c\tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = -\frac{e\tau}{m}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{24}$$

Inverting gives the current density:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau & 0 \\ \omega_c\tau & 1 & 0 \\ 0 & 0 & (1+(\omega_c\tau)^2)^{-1} \end{pmatrix} \tag{25}$$

In the high magnetic field limit $\omega_c\tau \gg 1$, and hence keeping only the dominant terms in the previous equation gives

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma_0}{\omega_c^2\tau^2}\begin{pmatrix} 0 & -\omega_c\tau & 0 \\ \omega_c\tau & 0 & 0 \\ 0 & 0 & (\omega_c\tau)^{-2} \end{pmatrix} \tag{26}$$

implying the only dominant terms are the off-diagonal ones. That is,

$$\sigma_{xy}=-\sigma_{yx}=-nec/B \tag{27}$$

and $\sigma_{xx}=\mathrm{O}\left(1/(w_c\tau)^2\right)$.